Algebraic theory of quadratic forms and Kaplansky's problem

Solutions (Exam)

Problem 1

1) a) Every positive real number is a square, so we get

$$\langle 1, 2, 3, -4\rangle \simeq \langle 1, 1, 1, -1\rangle$$

b) Note that the both quadratic forms are isotropic:

$q_1 = \langle 1, 1, -5\rangle$ and $q_2 = \langle \underbrace{2, -2}_{\substack{\simeq \\ \mathbb{H}}}, 5\rangle$

$1^2 + 2^2 - 5 \cdot 1 = 0$

We get $q_1 \simeq \mathbb{H} \perp \langle a\rangle$ and $q_2 \simeq \mathbb{H} \perp \langle b\rangle$

Note that $\det q_1 = \det q_2 = -5 \in F^\times/F^{\times 2}$

Therefore $a = b \in F^\times/F^{\times 2}$ and $q_1 \simeq q_2$

2) a) If $i(q) \geq 2$ we get $q \simeq \mathbb{H} \perp \mathbb{H} \perp q'$, where $\dim q' = 2$

Taking the determinant: $-1 = \det q = (-1)(-1)\det q' = \det q'$

$$\left.\begin{matrix}\det q' = -1 \\ \dim q' = 2\end{matrix}\right\} \Rightarrow q' \simeq \mathbb{H} \Rightarrow i(q) = 3.$$

b) Note that $i_1(q) = i(q_E)$ for some field ext. E/F

Note that $\det q_E = -1 \in E^\times/E^{\times 2}$ and $i(q_E) \geq 1$.

Assume $i(q_E) > 1$, then by (2) $i(q_E) = 3 = i_1(q)$

Then the higher Witt-indices of q = $\{\underset{i_0}{0}, \underset{i_1}{3}\}$, $i_0 < i_1$ $\Rightarrow ht\, q = 1$

$\Rightarrow$ By Lecture, we know that even dim. q.f. with $ht = 1$ is similar to a Pfister form, but then $\dim q = 2^k \neq 6$ and we get a contradiction.

Problem 2

1) Note that $\underset{\pi''}{9\langle 1\rangle \subseteq 16\langle 1\rangle = \langle 1,1\rangle^{\otimes 4}}$ Pfister form.

$9\langle 1\rangle$ is isotropic over $\mathbb{R}_9$ $\Rightarrow$ $\langle 1,1\rangle^{\otimes 4}$ is isotropic and hence hyperbolic $/\mathbb{R}_9$

We get $10\langle 1\rangle + 6\langle 1\rangle = 0$ in $W(\mathbb{R}_9)$

$\Rightarrow$ $10\langle 1\rangle = -6\langle 1\rangle$ in $W(\mathbb{R}_9)$ $(*)$

Claim: $6\langle 1\rangle$ is anisotropic $/\mathbb{R}_9$

If not, then $\underset{\text{isotropic}}{6\langle 1\rangle} \subseteq \underset{\text{hyperbolic}}{8\langle 1\rangle}$ Pfister

$\overset{\text{anis.}/\mathbb{R}}{8\langle 1\rangle}$ becomes hyperbolic over $\mathbb{R}_9 = \mathbb{R}[9\langle 1\rangle]$

By Lecture we know, that then $\dim 8\langle 1\rangle \geq \dim 9\langle 1\rangle$
Contradiction

Thus, it follows from $(*)$, that $\dim\left(10\langle 1\rangle_{\mathbb{R}_9}\right)_{an} = 6$

$\Rightarrow$ $i\left(10\langle 1\rangle_{\mathbb{R}_9}\right) = \dfrac{10-6}{2} = 2$.

2) Let $\varphi := n\langle 1\rangle$ and $q := (n-1)\langle 1\rangle$

Note that q is anisotropic $/\mathbb{R}$.
Assume the statement of the exercise is not true, then it ~~The statement of the exercise~~ can be formultated as follows:

$$\underset{x_1^2+\dots+x_n^2}{\varphi(x_1,\dots,x_n)} \in D_{\mathbb{R}(x_1,\dots,x_n)}(q)$$

By Subform Theorem, we get $\varphi \subseteq q$ $\Rightarrow$ $n = \dim\varphi \leq \dim q = n-1$

Contradiction.

Problem 3

1) ~~If~~ Case 1 : φ is isotropic (hence hyperbolic)

Then we can take for ρ any hyperbolic form of $\dim = 2^{m-n}$

Case 2 φ is anisotropic

Over the function field of π, φ becomes isotropic (and hence, as a Pfister form, hyperbolic)

Indeed, $\underset{\substack{\uparrow \\ \text{isotropic}}}{\pi_{F[\pi]}} \subset \underset{\substack{\uparrow \\ \text{isotropic}}}{\varphi_{F[\pi]}}$ $\Rightarrow$

~~By Lecture~~ It follows from the lecture (since π is a Pfister form) that $\varphi \simeq \pi \otimes \rho$ for some q.f. $\rho/_F$.

2) Case $m-n=1$

By 1) $\varphi \simeq \pi \otimes \rho$, where $\rho \simeq \langle a, b \rangle$ 2-dimensional

Note that $a \in D_F(\varphi)$ (indeed, $\varphi \simeq \underbrace{a\pi}_{\text{represents } a} \perp b\pi$)

$\| \leftarrow$ since φ is a Pfister form

$G_F(\varphi)$

Therefore, $\varphi \simeq a\varphi \simeq \pi \otimes \underbrace{\langle 1, ab \rangle}_{\pi' \text{ Pfister form.}}$

Induction on $m-n$.

By 1) we have $\varphi \simeq \pi \otimes \rho$, where $\rho \simeq \langle a, b \rangle \perp \rho'$

Same argument as in the case $m-n=1$ shows that $a \in D_F(\varphi) = G_F(\varphi)$

We get $\varphi \simeq a\varphi \simeq \pi \otimes (\langle 1, ab \rangle \perp q') \simeq \underbrace{\pi \otimes \langle 1, ab \rangle}_{\tilde{\pi}} \perp \pi \otimes q'$

$\Rightarrow \tilde{\pi} \subseteq \varphi \xRightarrow{\text{Induction}} \varphi \simeq \overbrace{\pi \otimes \langle 1, ab \rangle}^{\tilde{\pi}} \otimes \tilde{\tilde{\pi}}'$

Problem 4

1) We know from the lecture that any quaternion algebra is a central simple algebra and that the tensor product of c.s.a is once again a c.s.a.

2) Assume $\langle a, b\rangle \simeq \langle c, d\rangle$

By Lecture

$$\left(\frac{a,b}{F}\right) \simeq \left(\frac{c,d}{F}\right) \iff \underbrace{\langle 1,-a,-b,ab\rangle \simeq \langle 1,-c,-d,cd\rangle}_{\text{norm forms}}$$

We have

$$\langle 1\rangle \perp -\langle a,b\rangle \perp \langle ab\rangle$$
$$\wr\!\wr(1) \qquad \wr\!\wr(2)$$
$$\langle 1\rangle \perp -\langle c,d\rangle \perp \langle cd\rangle$$

(1) since $\langle a,b\rangle \simeq \langle c,d\rangle$

(2) since $ab = \det\langle a,b\rangle = \det\langle c,d\rangle = cd$ in $F^\times/F^{\times 2}$

3) ~~In the product~~ ~~We have~~ In the product from the def. of $s(q)$ and $s(q')$ we have the common part

$$A = \bigotimes_{3\le i<j\le n} \left(\frac{a_i, a_j}{F}\right)$$

and $s(q) \simeq B \otimes A$, $s(q') = C\otimes A$,

where
$$B = \left(\frac{a,b}{F}\right) \otimes \underbrace{\left(\frac{a,a_3}{F}\right)\otimes \dots \otimes \left(\frac{a,a_n}{F}\right)}_{\wr\wr\;\left(\frac{a,\, a_3\cdot\ldots\cdot a_n}{F}\right)\otimes M_{2^{n-2}}(F)} \otimes \underbrace{\left(\frac{b,a_3}{F}\right)\otimes \dots \otimes \left(\frac{b,a_n}{F}\right)}_{\wr\wr\;\left(\frac{b,\, a_3\cdot\ldots\cdot a_n}{F}\right)\otimes M_{2^{n-2}}(F)}$$

$$\simeq \left(\frac{a,b}{F}\right) \otimes \left(\frac{ab,\, a_3\cdot\ldots\cdot a_n}{F}\right) \otimes M_{2^{2n-3}}(F)$$

Similarly, $C = \left(\frac{c,d}{F}\right) \otimes \left(\frac{cd,\, a_3\cdot\ldots\cdot a_n}{F}\right) \otimes M_{2^{2n-3}}(F)$

By Witt cancelation Theorem, $q \simeq q' \Rightarrow \langle a,b\rangle \simeq \langle c,d\rangle$ ~~$\Leftrightarrow$~~ and $s(q) \simeq s(q')$

Then we have $ab = cd$ in $F^\times/F^{\times 2}$ and by 2) $\left(\frac{a,b}{F}\right) \simeq \left(\frac{c,d}{F}\right) \Rightarrow B \simeq C$

4) From the lecture we know that

$\exists$ a chain of isometries of diagonal forms

$$q = q_0 \simeq q_1 \simeq q_2 \simeq \dots \simeq q_m = q'$$

where q_i and q_{i+1} differs by at most two elements.

By question 3) we get $s(q_i) \simeq s(q_{i+1})$ $\forall\, i = 0, \dots, m$.

Therefore, $s(q) \simeq s(q')$.